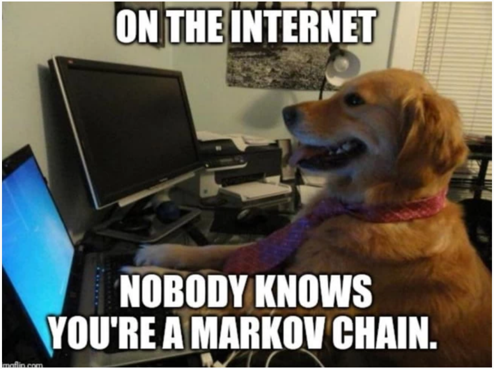
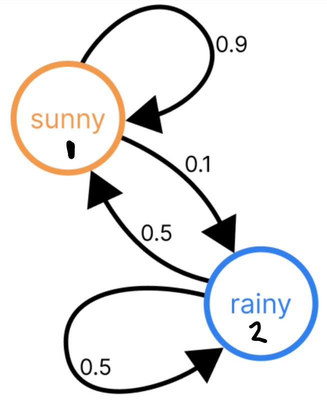
Games, graphs, and machines



August 27, 2024

Warm up

Write the transition matrix A for the following Markov chain.



$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{pmatrix}0.9 & 0.1\\ 0.5 & 0.5\end{pmatrix}$$

- Rows add up to 1
  ↳ "Stochastic".
- $\begin{pmatrix}0.9 & 0.1\\ 0.5 & 0.5\end{pmatrix}\underbrace{\begin{pmatrix}1\\1\end{pmatrix}} = \begin{pmatrix}1\\1\end{pmatrix}$

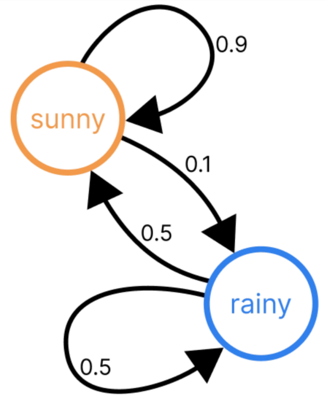
$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$

effectively summing rows

Warm up

Write the transition matrix A for the following Markov chain.

Calculate A^2. What do entries of A^2 represent?



$$\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.86 & 0.14 \\ 0.7 & 0.3 \end{pmatrix}$$

Two step prob.

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad A^2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A \cdot \underbrace{A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Why does kth power represent k-step probabilities?

$$A^2_{i,j} = A_{i,1} \cdot A_{1,j} + A_{i,2} \cdot A_{2,j}.$$

(Prob of $1 \to j$)

(Prob of $i \to 1$)

$\cdot$ = AND

$+$ = OR

sum

= Prob of going $i \to j$ in 2 steps.

Prob of taking $i \to 1 \to j$ } first term

Prob of taking $i \to 2 \to j$ } second term.

Large powers

same rows ⇒ eventual prob. indep of current state.

We have $A = EDE^{-1}$, where

$$E = \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & -1/6 \end{pmatrix}.$$

Find (approximate) A^k for large k. What do the entries represent?

$$A^2 = ED^2E^{-1} = (EDE^{-1})(EDE^{-1})$$

$$A^3 = ED^3E^{-1}$$

$$A^{1000} = E \cdot D^{1000} E^{-1}$$

eventual trans.

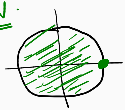
$$\lim_{k\to\infty} A^k = \begin{pmatrix} 1 & \cdot \\ 1 & \cdot \end{pmatrix}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\begin{pmatrix} 5/6 & 1/6 \\ \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5/6 & 1/6 \\ \cdot & \cdot \end{pmatrix} = \begin{array}{c} S \to \\ R \to \end{array}\begin{pmatrix} 5/6 & 1/6 \\ 5/6 & 1/6 \end{pmatrix}$$

When do large powers converge?

Suppose $A = EDE^{-1}$, where D is diagonal. When will A^k converge (to a matrix with finite entries) as k grows?

Conv.



Depends on $\underbrace{\text{entries}}_{}$ of D
eigenvalues

-1 ↓ osc.

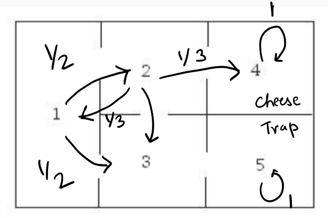
1 or abs < 1 $\Downarrow$ converge

is an entry > 1 (in abs) $\Rightarrow$ diverge

↑ Never happens for Markov chains.

A maze

A maze used for training rats has the following shape.



Suppose that at every stage, the rat picks a door at random (with equal probability) and goes through that door.* Write the Markov chain and the transition matrix.

* Except rooms 4 & 5 which contain cheese and a trap, respectively. Once there, the rat stays there forever.

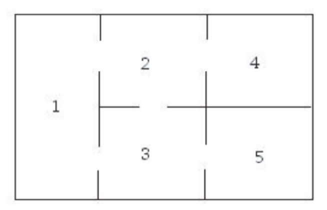
Powers of the maze

The matrix A is diagonalisable with eigenvalues $[-1/3, 1, 1, -0.43, 0.76]$.
Will powers of A converge? YES .

Powers of the maze

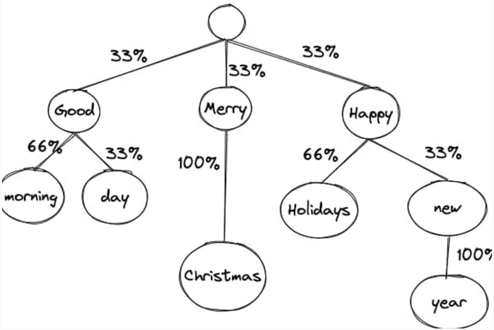
The powers of A converge to

$$\begin{pmatrix} 0 & 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 0.62 & 0.38 \\ 0 & 0 & 0 & 0.38 & 0.62 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$



Can you interpret the entries?

A text generator



Let A be the corresponding transition matrix. The powers of A stabilise. When do they stabilise? What is the first row of A^{100}?